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# Modelling the behaviour of homogeneous scalar turbulence

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The paper considers the problem of calculating the statistical characteristics of a passive scalar dispersed by a homogeneous turbulence field. In many turbulent shear flows the time-scale for the evolution of the scalar field is intrinsically related to that of the turbulent velocity field. This is by no means always the case, however, and it is at this more general situation that the present work is aimed. An approximate transport equation for the rate of dissipation of scalar variance is proposed which, it is argued, must contain (at least) two sink terms one of which responds to the time scale of the velocity field while the other reflects that of the scalar field itself. The model has been applied to the limited number of homogeneous scalar flows for which data are available and achieves satisfactory agreement as judged by the evolution of the mean-square scalar variance.

## 1. Introduction

Currently there is much activity in the development of second-order turbulence models. This appears to be the simplest closure level in which essential turbulent flow characteristics, e.g. transport, pressure-interactions, dissipation and effects of external force fields, can be directly incorporated. In second-order modelling, a truncated hierarchy of moment equations is used and closure is effected by expressing the unknown higher-order moments in terms of lower-order quantities. Rational closure approximations are best developed by considering a hierarchy of increasingly complex flows. In this way, the various physical features observed in turbulent flows can be considered individually, and thereby good representations for the higher-order terms associated with these phenomena may be developed.

In this paper, we present a second-order model which pertains to a homogeneous passive scalar field in decaying homogeneous turbulence without mean velocity gradients and spatially removed from solid boundaries. We consider both a scalar field without mean scalar gradients and one which contains a constant mean scalar gradient. These two flows contain features which are fundamental to scalar turbulence flows, and so our model should provide the basis for models describing more complex scalar

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flows. The main contribution of this paper lies in the provision of a prognostic equation for $\epsilon_c = \kappa \overline{c_{,j} c_{,j}}$, which represents half the molecular dissipation rate of scalar variance $\overline{c^2}$.† The equation itself is imbedded in the second-order closure scheme evolved by the present authors, Lumley & Newman (1977) and Launder (1976).

A number of workers (for example Spalding 1971) have presented scalar closure models in which $\epsilon_c$ is determined directly through explicit specification of a constant value for the ratio of mechanical to thermal time scales, $r \equiv (\overline{q^2}/\epsilon)/(\overline{c^2}/\epsilon_c)$, where $\epsilon \equiv \nu\overline{u_{i,j} u_{i,j}}$ is the dissipation rate of one half of the velocity variance, $q^2$ $(=\overline{u_i u_i})$. This procedure obviously provides a simpler route than through a prognostic equation. There is, however, no reason to expect that the time-scale ratio $r$ should exhibit a universal value in all scalar turbulent flows. Indeed, there are strong grounds for expecting *a priori* that $r$ should vary among differing types of scalar flows. The quantity $r$ expresses the ratio of local turnover times of the energy-containing velocity and scalar eddies; these energy-containing eddies are influenced significantly by the production mechanisms of the respective velocity and scalar fields. It appears reasonable therefore to expect that the level of $r$ will depend on these production mechanisms, and hence that $r$ might change among flows with differing influences of the production mechanisms. In fact, examination of existing scalar turbulence data supports this premise.

Warhaft & Lumley (1978) review the existing data concerning heated grid turbulence and present data from their studies of this type of flow. They note that the levels of $r$ vary (as a unique function of the relative positions of the peaks of the velocity and scalar energy spectra) from approximately 0·6 to 2·4 in the various grid experiments. Béguier, Dekeyser & Launder (1978) examined the data from a number of studies of thermal turbulence in thin shear flows (including boundary layer, pipe and wake flows) in which the production rates of both the velocity and scalar fields were roughly equal to the respective dissipation rates. They found the level of $r$ to be approximately 2·0 in all of the flows they investigated.‡ Finally, Launder (1975*a*) shows that a value of $r$ of 0·7–0·8 is needed to predict the strong rise of turbulent Prandtl number with Richardson number exhibited by Webster's (1964) data in a density stratified shear flow. The variations in $r$ displayed by these data preclude accurate prediction of these flows with a model in which $\epsilon_c$ is simply determined by way of a fixed value of $r$, a conclusion that has provided the impetus for the present work. It is found that all the available homogeneous scalar turbulence data are well simulated with a closure including the proposed prognostic equation for $\epsilon_c$.

The strategy pursued in devising the $\epsilon_c$ equation is in most respects an extrapolation of that adopted in Lumley & Newman (1977) and Launder (1976); the closure is developed in §3. However, before considering that problem in detail, we first examine, in §2, the equilibrium behaviour of the time-scale ratio $r_e$ in homogeneous scalar turbulence without mean scalar gradient.

† What really interests us is the rate at which scalar variance is fed into the low wavenumber end of the spectrum to begin its journey through the cascade to dissipation scales. In an equilibrium situation this is equal to the dissipation, and we will call it 'dissipation', but of course it is a quantity determined by the parameters of the energy-containing range; the fact that it can be written in terms of fluctuating velocity gradients is irrelevant. It might better be called spectral flux of scalar variance. See Tennekes & Lumley (1973).

‡ Note that the time-scale ratio $r$ in this paper is the reciprocal of that adopted by Béguier *et al.* (1978) and denoted by $R$.

## 2. Equilibrium decay levels of the time-scale ratio

We envisage an equilibrium decay regime to be a region of decaying scalar turbulence which is not significantly influenced by initial or boundary conditions. From a physical viewpoint, it seems plausible that, for such a flow with nearly isotropic scalar and velocity fields and with moderate to large turbulent Reynolds and Péclet numbers, the value of $r$ would be close to unity since, presumably, the energetic large-scale velocity eddies are those which most profoundly affect the energetic large-scale scalar eddies, distorting them on the scale of the large-scale velocity eddies. This large-scale distortion of the scalar field could be expected to force the scale size of the energetic scalar eddies to keep pace with that of the evolving energetic velocity eddies. In that event, the ratio $r_e$, viewed as the ratio of time-scales relevant to the large eddies, should be fairly close to unity.

Let us compare this conjectured development with the levels of $r$ found in a turbulent flow downstream from a heated grid where the mean temperature and velocity are uniform. In most of the heated grid studies (the only exception being that of Lin & Lin 1973) the levels of temperature fluctuations were kept sufficiently small for buoyancy effects to be negligible; the scalar contaminant, temperature, therefore behaved passively in these flows. The levels of anisotropy of the velocity fields, though not fully documented experimentally, were presumably small. On the other hand, values of the streamwise heat-flux correlation coefficients ranging from approximately 0·1 to 0·1 were measured in the various heated grid flows in which heat-flux measurements were performed, indicating the presence of 'anisotropy' in the scalar field in some of the flows. The heat-flux data are dealt with in detail in Warhaft & Lumley (1978), and we shall consider the heat-flux issue briefly in a later section. We mention here only that, at least qualitatively, the behaviour of the time-scale ratio in any of the documented grid turbulence flows appears to have been unaffected by the level of the heat-flux correlation coefficient in the flow. The data for the scalar and velocity variances for all the flows may be adequately represented with power-law expressions $\overline{q^2} \propto t^{-n}, \overline{c^2} \propto t^{-m}$; the time-scale ratio is obtained from the ratio of the exponents, $r = m/n$. Examination of the data reveals that in each documented grid flow the time-scale ratio remained at a constant level, within the accuracy of the data, over the entire observed streamwise extent of the flow (downstream of the point at which homogeneity was achieved). However, the level of $r$ varied among these flows from approximately 0·6 to 2·4. Warhaft & Lumley (1978) present strong evidence from their measurements that the level of $r$ in a heated grid turbulent flow may well be a unique function of the difference $k_v - k_c$, where $k_v$ and $k_c$ are the wave numbers corresponding to the peaks of the velocity and scalar energy spectra respectively. In decaying heated grid turbulence, the velocity and temperature spectra peak at successively lower wave numbers as the flow evolves downstream (Yeh & Van Atta 1973), because the smaller eddies in the flow decay more rapidly than the larger ones. However, Warhaft & Lumley (1978) found that the difference $k_v - k_c$ was approximately invariant with respect to downstream position in each of the flows they examined, and they suggest that the constancy of the level of $r$ in a heated grid flow may be attributable to this observation.

The uniformity of $r$ with respect to downstream position in heated grid turbulence does not conform to our conjectured picture of equilibrium decay. It is possible, however, that the high-Reynolds-number decay of thermal grid turbulence is not truly representative of an equilibrium flow. Indeed, the existing data suggest that the level of $r$ in heated grid turbulence may well be determined solely by the initial conditions (energy input scales) governing the flow, i.e. that the initial period of decay in thermal turbulence is not an equilibrium decay regime which is independent of the initial and boundary conditions. It is possible, of course, that cumulative large-scale velocity field distortions of the scalar field over a sufficiently long streamwise distance could eventually alter the spectral distribution of scalar energy to a form compatible with equilibrium decay with the velocity field. Some of the data, however, extend one turbulence decay time downstream of the grid (where we define the turbulence decay time $\tau$ by $d\tau = (\epsilon/\overline{q^2})\,dt$, where $t$ is real time); from what is known of spectral dynamics this should be long enough for any tendency towards an equilibrium decay to become apparent. Further studies of thermal grid turbulence (at larger Reynolds numbers) with longer tunnel sections might help to resolve these matters.

We may further compare the notion of equilibrium decay with existing theoretical analyses of the decay of concomitant isotropic scalar and velocity fields which provide input regarding $r_e$. In a recent investigation, Newman & Herring (1979) applied the Test Field Model (Kraichnan 1971) to the study of an isotropic passive scalar in an isotropic velocity field. The Test Field Model simulations of scalar turbulence performed in this study extend over roughly 1·5 turbulence decay times and exhibit approximate self-preservation of the velocity and scalar energy dissipation and transfer spectra. Model predictions for scalar and velocity dissipation spectra for large turbulent Reynolds and Péclet numbers show good agreement with the atmospheric data of Champagne *et al.* (1977), after intrinsic model scale factors (which regulate the build-up of triple moments) were fitted by requiring agreement between predicted and empirical velocity and scalar energy spectra in the inertial wavenumber range. Model predictions of self-preserving scalar and kinetic energy, dissipation and transfer spectra show moderate agreement with the heated grid turbulence data of Yeh & Van Atta (1973). Newman & Herring (1979) also presented the temporal evolution of the time-scale ratio and of the normalized decay rates $\psi \equiv \dot{\epsilon}q^2/\epsilon^2$ and $\psi_c \equiv \dot{\epsilon}_c\overline{c^2}/\epsilon^2$. In all the simulations the self-preserving state was one in which $\overline{q^2}$ and $\overline{c^2}$ underwent asymptotically a power-law decay with a decay exponent of nearly unity. Consequently, the time-scale ratio in each of the simulations approached a value of about unity and both $\psi$ and $\psi_c$ asymptoted to levels approximately equal to 4·0. This asymptotic state is consistent with our physical picture of an equilibrium decay flow. We note (cf. Lumley & Newman 1977) that the level of $\psi$ in nearly isotropic turbulence varies as a function of turbulence Reynolds number $R_l$ defined below, but the dependence is weak and, in fact, the Test Field Model value of $\psi \sim 4{\cdot}0$ is within 11 % of the values of $\psi$ appropriate for grid turbulence over the observed range of $R_l \geqslant 100$. Moreover, in the self-preserving state in each of the simulations, the velocity and scalar energy spectra peak at successively lower wavenumbers as time increases, and the velocity spectrum peaks at a wavenumber approximately twice that of the scalar spectrum. This spectral-peak difference is in fair agreement with the values found in heated grid data for the cases where $r = 1$ (Yeh & Van Atta 1973; Warhaft & Lumley 1978). All the above behaviours are independent of Reynolds and Prandtl (Schmidt) numbers over the range covered by the simulations, thus indicating an insensitivity of the model to changes in the levels of the molecular diffusivities. Newman & Herring (1979) note that this independence, if observed, would probably be exhibited in real turbulence only for cases of moderate to large Reynolds and Péclet numbers (where $\epsilon$ and $\epsilon_c$ would be approximately independent of the levels of the diffusivities – see Tennekes & Lumley 1973).

We infer from the above that the scalar Test Field Model simulations provide fairly good agreement with existing experimental data of grid turbulence provided the initial value of $r$ is near unity. For $r \neq 1$, however, the data show that $\psi_c$ and $r$ remain at their initial levels over evolutionary periods up to $\Delta\tau \approx 1$, while the simulations predict an asymptotic approach from the initial levels to $\psi_c \approx 4{\cdot}0$, $r = 1{\cdot}0$ within this period. As Newman & Herring (1979) remark, the Test Field simulations start from specified initial spectral forms with zero transfer spectra, whereas the grid turbulence fields evolve from coalescing heated wakes and hence exhibit significantly different initial energy and transfer spectra. Possibly the experimental evolution of $r$ and $\psi_c$ might be better reproduced by the simulations if the initial spectral forms for the latter could be set to agree better with the data spectra. Such an investigation would entail alteration of the existing theory prescriptions, but might well serve to shed some light on the issues.

We note in passing the results of other analyses of decaying isotropic scalar turbulence that are based on similarity assumptions; the various similarity theories are discussed in detail by Hinze (1975) and Monin & Yaglom (1975). Generally, these prescribe either partial or complete self preservation of the scalar and velocity spectra consistent with certain integral invariants for the flow (e.g. Loitsianskii 1939; Corrsin 1951; Saffman 1967). The imposed conditions serve to determine decay laws for $\overline{q^2}$ and $\overline{c^2}$ from which values for $r$ are readily determined. Levels of $r$ from the various similarity theories are scattered about a value of 1·0 ranging from about 0·5 to 1·5. The validity of the assumptions in these descriptions is not substantiated, however, and so we shall not employ their results in our work.

Finally, we consider two exact results for scalar turbulence decay which might be viewed as special cases of equilibrium decay of homogeneous scalar turbulence without mean scalar and velocity gradients. The first is the final period of decay for small Reynolds and Péclet numbers. For the final period the non-linear terms in the scalar and velocity equations of motion may be neglected, and the resulting equations solved exactly (Corrsin 1951). The solutions for $\overline{q^2}$ and $\overline{c^2}$ are power-law decays with decay exponents of $\frac{5}{2}$ and $\frac{3}{2}$ respectively. This gives $r = \frac{3}{5}$ as the appropriate value for the final period of decay. This value holds for all levels of anisotropy of the velocity and scalar fields.

The second case is that of one-dimensional scalar turbulence. The velocity field may be solved exactly for this case (Lumley & Newman 1977), because the nonlinear terms in the velocity equation of motion vanish. The solution for $\overline{q^2}$ is again a power law with a decay exponent of 2. On the other hand, the solution for $\overline{c^2}$ for one-dimensional turbulence is not easily obtained in general, because the convective term containing the non-zero fluctuating velocity component does not vanish. However, this nonlinear term does vanish for the case of one-dimensional turbulence in which the fluctuating scalar field is independent of position in the direction of the fluctuating velocity component. The scalar transport equation for this flow has the same form as that for the final period of decay and hence yields a power law solution for $\overline{c^2}$; however, the decay exponent is now unity, instead of $\frac{3}{2}$, owing to the one-dimensional nature of the spectrum. Thus, for this latter case, the value of the time-scale ratio is 0·5. Lumley & Newman (1977) argue that for this flow their solution for the decay of $\overline{q^2}$ is, in fact, a singular-limit solution which is achieved only for one-dimensional turbulence, and that the final-period decay solution for $\overline{q^2}$ is appropriate for all quasi-one-dimensional flows right up to the point of one-dimensionality. They therefore adopted the value of the final-period decay exponent for $\overline{q^2}$ as the value for the limit of one-dimensional turbulence. If we proceed in an analogous fashion with the scalar decay problem (although there is no strict analogy here) we obtain $r = 0{\cdot}6$ (the final-period value) for the case of one-dimensional scalar turbulence.

## 3. A model for the scalar dissipation equation in homogeneous turbulence

### 3.1. *Preliminaries*

The exact transport equations describing the development of the intensity of scalar fluctuations in a homogeneous turbulent flow without mean velocity may be written (Lumley 1972; Bradshaw 1976) as

$$d\overline{c^2}/dt = -2\overline{u_j c}\,C_{,j} - 2\kappa\overline{c_{,j}c_{,j}} = -2\overline{u_j c}\,C_{,j} - 2\epsilon_c, \tag{3.1}$$

$$d\epsilon_c/dt = -2\kappa\overline{c_{,i}u_{j,i}}\,C_{,j} - 2\kappa\overline{c_{,j}c_{,i}u_{i,j}} - 2\kappa^2\overline{c_{,ij}c_{ij}}, \tag{3.2}$$

$$d\overline{u_i c}/dt = -\overline{u_i u_j}\,C_{,j} - \beta_i\overline{c^2} - (\kappa+\nu)\overline{c_{,j}u_{i,j}} - \overline{p_{,i}c}/\rho, \tag{3.3}$$

$$d\overline{u_i u_j}/dt = -(\beta_j\overline{u_i c} + \beta_i\overline{u_j c}) - (2\nu\overline{u_{i,k}u_{j,k}} - \tfrac{2}{3}\epsilon\delta_{ij} + \overline{p_{,i}u_j}/\rho + \overline{p_{,j}u_i}/\rho) - \tfrac{2}{3}\epsilon\delta_{ij}, \tag{3.4}$$

$$d\epsilon/dt = -2\nu\overline{u_{i,k}u_{i,j}u_{j,k}} - 2\nu^2\overline{u_{i,jk}u_{i,jk}} - 2\nu\beta_i\overline{u_{i,j}c_{j}}. \tag{3.5}$$

In the above set $\kappa$ and $\nu$ represent the thermal diffusivity and kinematic viscosity, $p$ denotes the instantaneous fluctuation in pressure about the mean value, $\rho$ represents the fluid density while $\beta_i$ is a component of the buoyancy vector whose magnitude is $\alpha g/c$ ($\alpha$ being the dimensionless volumetric expansion coefficient and $g$ the gravitational acceleration). The decomposition of the Reynolds stress equation (3.4) is that proposed in Lumley & Newman (1977); the form given serves to separate the diagonal and off-diagonal elements of the dissipation tensor $\nu\overline{u_{i,k}u_{j,k}}$, which is observed to become increasingly isotropic with increasing Reynolds number (Batchelor 1953). The mean scalar gradient $C_{,j}$ is, by virtue of the requirement of homogeneity, understood to be uniform over the flow field.

In what follows the main aim is to devise a rational closure of equation (3.2) that faithfully imitates the behaviour of $\epsilon_c$. The turbulent velocity field is considered to be adequately described by the recent proposals of Lumley & Newman (1977) for closing (3.4) and (3.5) for cases in which buoyant influences are negligible. These equations are represented as:

$$d\overline{u_i u_j}/dt = -\epsilon\phi_{ij} - \tfrac{2}{3}\epsilon\delta_{ij}, \tag{3.6}$$

$$d\epsilon/dt = -\psi\epsilon^2/\overline{q^2}, \tag{3.7}$$

where

$$\psi = \tfrac{14}{5} + 0{\cdot}98\exp(-2{\cdot}83R_l^{-\frac{1}{2}})\{1 - 0{\cdot}337\ln(1 + 27{\cdot}5/\mathrm{II})\}$$

$$\phi_{ij} = \tilde{\phi}_{ij} - \tfrac{1}{3}\tilde{\phi}_{pp}\,\delta_{ij}, \quad \tilde{\phi}_{ij} = \sum_{\alpha=1}^{3} g(b_{(\alpha)} - \tfrac{1}{6})\,x_i^{(\alpha)}x_j^{(\alpha)}$$

$$g(x) = 2x + \exp(-7{\cdot}77R_l^{-\frac{1}{2}})\{0{\cdot}633(4\sin 2\pi x - \sin 4\pi x + \sin 6\pi x) + 1{\cdot}70R_l^{-\frac{1}{2}}(1{\cdot}26\sin 2\pi x - 0{\cdot}112\sin 4\pi x)\}.$$

The quantities $b_{(\alpha)}$ and $x_i^{(\alpha)}$ are respectively the eigenvalues and eigenvectors of $b_{ij}$ where $b_{ij} = \overline{u_i u_j}/\overline{q^2} - \frac{1}{3}\delta_{ij}$ is a tensor measure of the anisotropy of the velocity field and $\text{II} = b_{ij} b_{ij}$; $R_l = (\overline{q^2})^2/9\epsilon\nu$ is the turbulence Reynolds number.

For the scalar-flux equations, we shall also, with certain exceptions, adopt closures that have been proposed and utilized elsewhere. In §3.2 we consider the closure of the $\epsilon_c$ equation for the simplest case in which mean temperature gradients and buoyant effects are absent (and for which experimental data are more plentiful). Subsequently, the effects produced by these two additional processes are considered.

### 3.2. *Flows without mean scalar gradients*

The equations for the scalar variance and (half) its dissipation rate for a homogeneous scalar turbulent flow with scalar fluxes but without mean scalar gradients or buoyancy effects are given by

$$d\overline{c^2}/dt = -2\epsilon_c, \tag{3.8}$$

$$d\epsilon_c/dt = -2\kappa\overline{c_{,i}c_{,j}u_{i,j}} - 2\kappa^2\overline{c_{,j}c_{,ij}}. \tag{3.9}$$

By analogy with (3.7), we rewrite (3.9) as

$$d\epsilon_c/dt = -\psi_c \epsilon_c^2/\overline{c^2}, \tag{3.10}$$

thereby shifting attention to the dimensionless decay rate $\psi_c$ for which we must develop a model. We shall employ the invariant modelling technique after Lumley (1970*a*).

We observe that we could determine $\psi_c$ uniquely from equations (3.8), and the corresponding velocity field and scalar flux equations given above as a functional

$$\psi_c(\overline{u_i u_j}, \overline{c^2}, \overline{u_i c}, \epsilon, \epsilon_c, \kappa, \nu), \tag{3.11}$$

(where the functional dependence may range over the entire flow history) if we knew the time histories of the arguments. We have included the scalar flux $\overline{u_i c}$ in (3.11) with the view that the presence of scalar fluxes in a homogeneous scalar flow might affect the evolution of $\epsilon_c$. Next, if we presume that changes in the mean flow are slow relative to turbulence memory times (as discussed in Lumley & Newman 1977) the functional (3.11) may be evaluated simply as a function of the included arguments at the current time. We may now use dimensional analysis to group the arguments of $\psi_c$. Assuming $\kappa \cong \nu$, $\psi_c$ becomes a dimensionless function of 13 independent quantities which contain the dimensions of temperature, time and length, and we may therefore form 10 independent dimensionless groups. If we choose the anisotropy tensor $b_{ij}$ (which has 5 independent components since its trace vanishes), the time-scale ratio, the turbulence Reynolds number and a scalar-flux 'anisotropy' tensor defined as

$$f_i = \overline{u_i c}/(\overline{q^2}\,\overline{c^2})^{\frac{1}{2}}, \tag{3.12}$$

and then require invariance of $\psi_c$ under general coordinate transformations (see Lumley 1970*b*), we may represent $\psi_c$ as a function of tensor invariants. Retention of invariants to second order in anisotropy of the velocity and scalar fields yields

$$\psi_c = \psi_c(\text{II}, r, R_l, \text{II}_c) \tag{3.13}$$

where $\text{II}_c = f_i f_i$. Here we have excluded the higher-order invariant quantities on the grounds that: (i) we shall be dealing with, at most, only moderately anisotropic turbulence (in which the higher-order invariants should exert only a weak effect) and

(ii) there are insufficient sets of experimental data to determine the functional dependence of $\psi_c$ upon the higher-order invariants even if the desirability of their inclusion were acknowledged.

The form of the function $\psi_c$ must now be determined through consideration of existing data and theoretical results concerning decay of homogeneous scalar turbulence. We consider the influence of $\mathrm{II}_c$ upon $\psi_c$ first. The only studies of the decay of a homogeneous scalar field in which scalar fluxes as well as variances were measured are those of Mills *et al.* (1958), Yeh & Van Atta (1973) and Warhaft & Lumley (1978). Now, it is easily deducible from (3.8) and (3.10) that if, locally, the scalar variance is decaying at a rate proportional to $t^{-a}$ then the quantity $\psi_c$ is given by $2(1+a^{-1})$ independent of a possible virtual origin for the local scalar fluctuations. There is thus a readily apparent connection between the rate of decay of scalar variance and $\psi_c$. The levels of scalar-flux anisotropies for the above three studies, however, show apparently no connection with the corresponding values of $\psi_c$. Consequently, we have adopted the view that the parameter $\mathrm{II}_c$ must be excluded from (3.13), an important simplification.

We must now develop a form for $\psi_c = \psi_c(\mathrm{II}, r, R_l)$. Unfortunately, the ranges of the values of the arguments II and $R_l$ observed in the homogeneous scalar turbulence data are fairly limited. Therefore, we shall approach the problem by directly modelling the behaviour of $r$ from measurements and theory. We note here that the presence of the time-scale ratio in the argument set for $\psi_c$ provides a means of coupling the scalar and velocity fields in a manner which ensures compatible evolution (e.g. non-divergent growth of the scalar and velocity time scales, which are defined respectively as $T_c = \overline{c^2}/\epsilon_c$, $T_q = \overline{q^2}/\epsilon$) of concomitant scalar and velocity fields in scalar decay simulations.

Now, in order to develop a form for $\psi_c$, we have found it convenient to presume that deviations of $r$ from its equilibrium value $r_e$ are small. This presumption, like many others in second-order modelling, is often at variance with observations; however, it allows us to obtain an expression for $\psi_c$ which provides for satisfactory simulation of the evolution of the scalar field and thus of $r$ itself, in decaying scalar turbulence. Assuming that deviations are small, we may expand $\psi_c$ in a Taylor series in $r^{-1}-r_e^{-1}$ and truncate at the first order. If we then absorb the term $r_e^{-1}$ into the 'constant' term, we obtain

$$\psi_c = Br^{-1}+D, \tag{3.14}$$

where $B$ and $D$ are functions of II, $r_e$ and $R_l$ in general. With this linear expansion, the scalar dissipation equation (3.7) now becomes

$$\frac{d\epsilon_c}{dt} = -B\frac{\epsilon\epsilon_c}{\overline{q^2}} - D\frac{\epsilon_c^2}{\overline{c^2}} = -\epsilon_c\left(\frac{B}{T_q}+\frac{D}{T_c}\right). \tag{3.15}$$

We observe that realizability of $\epsilon_c$ (Schumann 1977) is automatically guaranteed with this form for the $\epsilon_c$ equation. We may employ equation (3.15) together with those for $\overline{c^2}$, $\overline{q^2}$ and $\epsilon$ to form an equation for the time scale ratio in terms of the decay time scale $\tau$ defined previously. We obtain

$$\frac{1}{r}\frac{dr}{d\tau} = (\psi - B - 2) + r(2-D). \tag{3.16}$$

We may now parameterize the coefficient functions $B$ and $D$ in (3.16) in a manner which ensures that $r$ behaves properly in simulations. We first summarize our views regarding the behaviour of $r$ in decaying scalar turbulence which derive from the discussions in §2 above.

We shall adopt the view that a decaying homogeneous scalar turbulent flow will exhibit equilibrium decay for all time after cumulative distorting effects of the velocity field redistribute the initial spectral distribution of scalar energy into a distribution compatible with equilibrium decay with the velocity field. We envisage equilibrium decay flow to exhibit levels of $r = 1$ at moderate and large values of $R_l$ and small levels of II (as indicated by the Test Field Model simulations). Further, we shall adopt the final period value $r = 0{\cdot}6$ as the appropriate equilibrium value for decay at very low levels of $R_l$ at all levels of II ($0 \leqslant \mathrm{II} \leqslant \frac{2}{3}$). In addition, we shall choose the final period value for $r$ for the equilibrium decay case of one-dimensional turbulence with an arbitrary spatial distribution of the scalar field. It is convenient for our modelling purposes to represent this equilibrium decay behaviour in a parameterized form for the equilibrium time-scale ratio $r_e$. Since we lack further information regarding $r_e$, the simplest approach appears to be that of adopting the forms of the $R_l$ and II expressions in the parameterization for $\psi$ presented by Lumley & Newman (1977). With these expressions we obtain

$$r_c = 0{\cdot}6 + 0{\cdot}4 \exp(-2{\cdot}83 R_l^{-\frac{1}{2}})\{1 - 0{\cdot}337 \ln(1 + 27{\cdot}5\,\mathrm{II})\} \tag{3.17}$$

which yields our adopted values for $r_e$ for the three limiting cases of turbulence considered here. Finally, we adopt the view that the approach to equilibrium decay from arbitrary initial conditions is very slow, as exhibited in the heated grid data.

Since $r_e$ represents an 'asymptotic' level for $r$ for decaying scalar turbulence, we require that

$$dr/d\tau = 0 \quad \text{at} \quad r = r_e, \tag{3.18}$$

which implies that

$$\psi = B + 2 + r_e(D - 2). \tag{3.19}$$

Using this result, we may rewrite (3.16) as

$$\frac{1}{r}\frac{dr}{d\tau} = (r - r_e)(2 - D). \tag{3.20}$$

In a strict sense, we should view $r$ as a function of $\tau$, II and $R_l$ and regard the derivative in (3.18) as a partial derivative with respect to $\tau$ with II and $R_l$ (and hence $r_e$) held at fixed values. However, we have found the condition (3.18) as stated both convenient and adequate for our modelling purposes. If we now presume that, at least locally, $r$ should approach $r_e$ monotonically, then (3.20) requires that $D \geqslant 2$ for all II and $R_l$. In addition, if we require that $B \geqslant 0$, which ensures that both terms in the $\epsilon_c$ equation will act to diminish $\epsilon_c$ in homogeneous decaying scalar flows, then with (2.19) we find that $D \leqslant \frac{10}{3}$. Thus $D$ lies in the fairly narrow band $2 \leqslant D \leqslant \frac{10}{3}$.

It remains to determine a form for $D$. We note from (3.20) that we may adjust the rate of evolution of $r$ through our specification of $D$. Unfortunately, the existing nearly isotropic heated grid data are insufficient to determine the influence of II, $r_e$ or $R_l$ on the evolution rate of $r$. Certainly it seems possible that the rate of change of $r$ could vary with these parameters, but further experiments are needed to confirm this. The most that can be inferred from the data is that the rate of evolution of $r$ is slow. Lacking further information we have chosen simply to specify a constant value for $D$ which is slightly in excess of 2 so that $r$ will display a slow approach to its equilibrium value. In

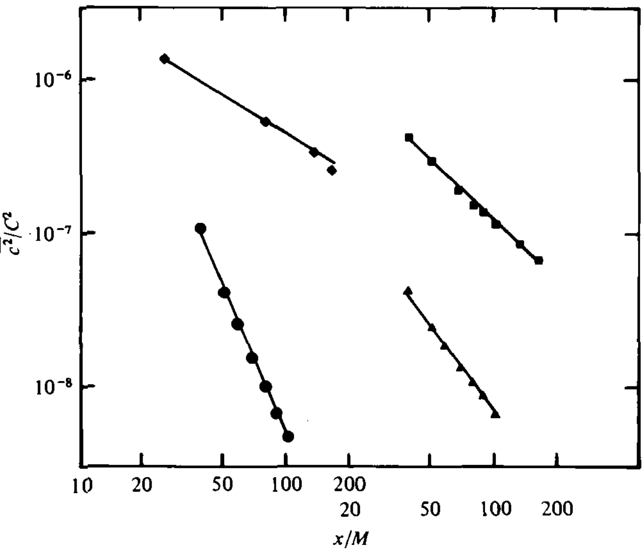


FIGURE 1. Decay of mean-square temperature variance in grid turbulence. Symbols, experiment of Warhaft & Lumley (1978); lines, prediction with present model.

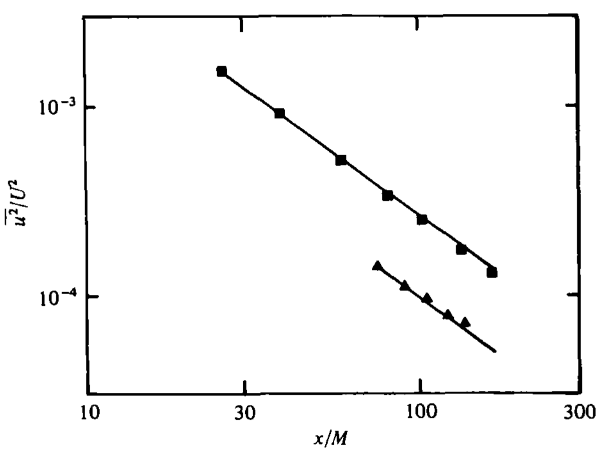


FIGURE 2. Decay of streamwise velocity fluctuations in grid turbulence. ■, Warhaft & Lumley (1978); ▲, Alexopoulos & Keffer (1971); — present predictions.

fact $D$ was specified as 2·02 so that, for the largest of the measured levels of $r$ in grid turbulence, 5 % of the excess of $r$ above its equilibrium level disappeared after a time interval $\Delta\tau$ of unity. (The choice of 5 % was arbitrary but produced results within the data scatter.) The functional variation of $B$ is now fixed through equation (3.19). We note that $B$ varies from 1·76 to 0·79 as $R_l$ and II range over their allowed values. For moderate to large levels of $R_l$ and small to moderate levels of II, $B$ takes values from about 1·4 to 1·76, so that, for this fairly general case, the two terms in the $\epsilon_c$ equation (i.e. those with coefficients $B$ and $D$) carry about equal weight provided $r$ is of order unity. The variability of $B$ with $R_l$ and II ensures the desired behaviour of $r$ in simulations.

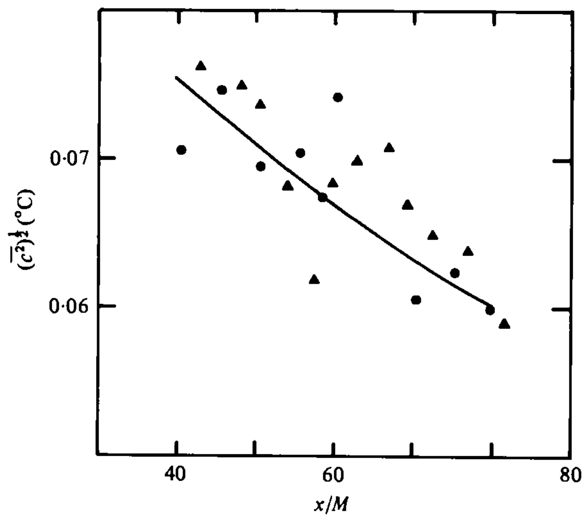


FIGURE 3. Decay of root-mean-square temperature variance. Symbols, experiments of Mills & Corrsin (1959); line, present predictions.

The ability of the above model to simulate the behaviour of heated grid turbulence may be assessed from figures 1–3. In these figures the points represent data values while the continuous lines represent model predictions. (Starting values of $\epsilon$ and $\epsilon_c$ in the predictions were chosen to give the correct initial slope to the experimental curves of $\overline{q^2}$ and $\overline{c^2}$.) Figure 1 provides a comparison of the calculated and measured decay of $\overline{c^2}/C^2$ for tests reported by Warhaft & Lumley (1978). These measurements cover a greater range of downstream evolution than alternative data. In all cases good agreement between the experimental and predicted values is obtained. For completeness the corresponding measurements and prediction of the decay of the streamwise velocity fluctuations are shown in figure 2; again satisfactory agreement is obtained. Finally, in figure 3 we present the data and prediction of the decay of $(\overline{c^2})^{\frac{1}{2}}$ for the flow investigated by Mills & Corrsin (1959). This flow is of particular interest because it is the only heated grid flow in which the decay begins with a strongly anisotropic velocity field (the prediction of the decaying velocity field is given by Lumley & Newman, 1977 and so is omitted here). We observe from the figure that the temperature decay for this flow is predicted within the experimental scatter.

### 3.3. *Flows with a mean scalar gradient*

In this section the range of applicability of the $\epsilon_c$ equation is extended to include homogeneous flows with a constant mean scalar gradient. The exact equation set describing this situation is (3.1–3.5). Buoyancy terms in the scalar flux and Reynolds stress equations will be retained here, because for the experiment to be considered (that of Alexopoulos & Keffer 1971) these terms, while relatively small, are not entirely negligible. It is convenient to write these equations in parameterized form as

$$d\overline{u_i c}/dt = +\phi_i^q - C_{,j}\overline{u_i u_j} - \beta_i \overline{c^2}, \tag{3.21}$$

$$d\overline{u_i u_j}/dt = +\phi_{ij}^q - \tfrac{2}{3}\epsilon\delta_{ij} - (\beta_j \overline{u_i c} + \beta_i \overline{u_j c}), \tag{3.22}$$

where $\phi_i^q$ and $\phi_{ij}^q$ represent closure models for the pressure gradient-velocity and pressure gradient-scalar correlations augmented by viscous terms which vanish at high Reynolds and Péclet numbers. The forms adopted here are

$$\phi_{ij}^q = -\epsilon\phi_{ij} + 0{\cdot}3\,(\delta_{qi}\overline{u_j c} + \delta_{qj}\overline{u_i c} - \tfrac{2}{3}\delta_{ij}\overline{u_q c})\,\beta_q, \tag{3.23}$$

$$\phi_i^q = -G\frac{\epsilon}{q^2}\overline{u_i c} + \tfrac{1}{3}\beta_i\overline{c^2}. \tag{3.24}$$

The terms containing the buoyancy vector represent the effects of gravity on the fluctuating pressure field, the forms given being exact in the case of isotropic turbulence (Lumley 1975; Launder 1975*b*). The first term in $\phi_i^q$ is the return-to-isotropy part of the pressure gradient-scalar correlation. The form proposed here has been employed in recent studies by Zeman & Lumley (1976) and by Launder (1976). We have chosen a value of 6·6 for the constant $G$ which (as described below) provides the best agreement with the data of Alexopoulos & Keffer (1971). Our value differs by no more than 12 % from the corresponding values used by Zeman & Lumley (1976) and by Launder (1976) to predict heat transport in the presence of substantial mean shear and substantial anisotropy of the fluctuating velocity field. The fairly uniform levels of $G$ obtained in these three studies encourage the view that a constant value for $B$ may be usefully adopted to predict a significant range of flows.

We may proceed to develop a closure for the $\epsilon_c$ equation in an analogous fashion to that of § 3.2. If we apply the invariant modelling technique to the $\epsilon_c$ equation in the set (3.1–3.5) then we may write

$$\dot{\epsilon}_c = -\psi_c^q\epsilon_c^2/\overline{c^2}, \tag{3.25}$$

where, in general, $\psi_c^q$ may be represented as an invariant function of the tensor invariants introduced in § 3.2 plus additional invariants formed with the buoyancy vector and the mean scalar gradient vector. Unfortunately, there are many more arguments of $\psi_c^q$ than there are documented flows with which to evaluate their importance. In fact, we have only the measurements of Alèxopoulos & Keffer (1971) with which to determine the influence of a mean scalar gradient on the evolution of $\epsilon_c$ in homogeneous scalar turbulence. In this experiment, a virtually linear cross-stream temperature gradient in a homogeneous decaying turbulent velocity field (with moderate $R_l$ and low II values) was established by means of a selectively heated grid.† To model this type of flow we must include a term (or terms) in our representation for $\psi_c^q$ which provide for production of $\epsilon_c$. In this manner, we may ensure that $\epsilon_c$ will keep pace with $\overline{c^2}$ (which will be fed by gradient production). On the other hand, the low levels of II in the data of Alexopoulus & Keffer (1971) preclude an accurate determination of the possible influence of anisotropy in a closure for $\psi_c^q$. Consequently, we shall exclude all of the invariants in the argument list of $\psi_c^q$ which are tensor products of $b_{ij}$ with the scalar field vectors. Moreover, we exclude buoyancy effects in our closure for the $\epsilon_c$ equation since the existing data do not warrant their inclusion. If terms up to second order in the remaining arguments of $\psi_c^q$ are retained, we obtain

$$\psi_c^q = \psi_c^q(\mathrm{II}, r, R_l, t_i t_i, t_i f_i), \tag{3.26}$$

† Wiskind (1962) had made an earlier study of this flow though his results are less complete and show more scatter than the Alexopoulos–Keffer data.

where $t_i$ is the dimensionless mean scalar gradient $C_{,i}(\overline{q^2}\overline{c^2})^{\frac{1}{2}}/\epsilon_c$. We shall adopt a form for $\psi_c^g$ through consideration of these arguments. The parameter $\text{II}_c$ has been omitted since, as noted in the previous section, decaying scalar grid turbulence shows no sensitivity to this quantity.

The two dimensionless parameters $t_i t_i$ and $t_i f_i$ could both be utilized in a closure for the $\epsilon_c$ equation to provide for production of $\epsilon_c$ in simulations of flows with mean scalar gradients. On the other hand, these two parameters vary in a closely similar way across many free shear flows, since the effective diffusivity is often nearly uniform at any section. It would therefore be gratuitous to retain both terms – at least until further experimental data are obtained. We retain the term $t_i f_i$ which represents (the negative of) the ratio of the production to dissipation rates of $\overline{c^2}$.

We have no other data with which to guide our modelling of $\psi_c^2$; however, we may consider the results of two analytical calculations which pertain to homogeneous scalar turbulence with an imposed constant mean scalar gradient. The first result concerns the final period of decay at small Reynolds and Péclet numbers. The equation of motion for the fluctuating scalar field contains two nonlinear terms for this type of flow: $c_{,j} u_j$ and $C_{,j} u_j$. However, for the final period the dissipation length scale is comparable to the integral length scale, and the ratio of these two scales is invariant with time. Consequently, the two non-linear terms should be of comparable magnitude during the final period, and thus may both be neglected in the scalar field equation. The solution for $\overline{c^2}$ will thus be identical to the final period case without a mean scalar gradient considered in §2, and therefore the time-scale ratio will again have the value 3/5 for all levels of anisotropy of the velocity and scalar fields.

The second result pertains to one-dimensional scalar turbulence with an imposed constant mean scalar gradient which is aligned in the direction of the fluctuating velocity component. The scalar field equation of motion for this type of turbulence is non-linear in general; however, the equation is linear in the fluctuating scalar and velocity variables for the case in which the fluctuating scalar field is independent of position in the direction of the fluctuating velocity. We may therefore readily obtain a closed form solution for $\overline{c^2}$ and $\epsilon_c$ for this latter case. We find that $\overline{c^2}/\epsilon_c = t$ (as is also true for $\overline{q^2}/\epsilon$ in this flow) so that the time-scale ratio is equal to unity in this case. Moreover, if (as in §2) we regard this case of one-dimensional turbulence as a singular limit, this final period solution will be valid at *all* values of Reynolds number.

It now remains to specify a form for $\psi_c^g$ which is consistent with our rather limited knowledge of the behaviour of $\epsilon_c$. The simplest satisfactory representation for $\psi_c^g$ appears to be to add a 'production' term to the homogeneous scalar decay closure

$$\psi_c^g = \psi_c - E t_i f_i, \tag{3.27}$$

(where $E$ is a constant) which upon substitution into (3.25) yields the following dissipation equation:

$$\boxed{\dot{\epsilon}_c = -B\frac{\epsilon\epsilon_c}{\overline{q^2}} - D\frac{\epsilon_c^2}{\overline{c^2}} + E\overline{u_i c}C_{,i}\frac{\epsilon_c}{\overline{c^2}}} \tag{3.28}$$

We note that the final term in (3.28) retains the realizability property exhibited by our homogeneous scalar decay model. A value of $-1{\cdot}955$ has been chosen for $E$ (with $G = 6{\cdot}6$) by minimizing the mean-square error between measurements and predicted values (using equations (3.28), (3.21), (3.7) and (3.21)–(3.24)) of the Alexopoulos & Keffer (1971) flow.

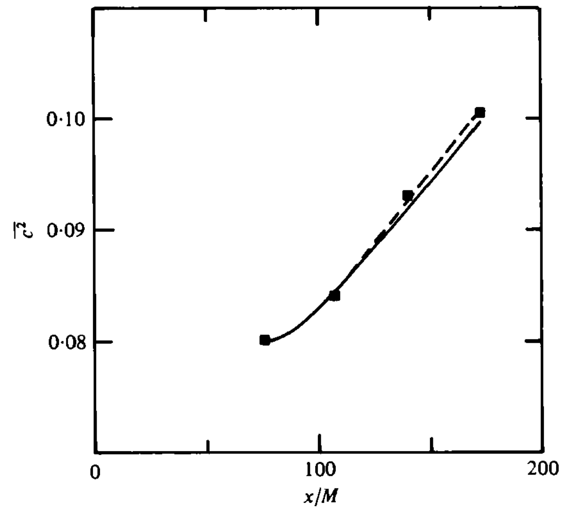


FIGURE 4. Mean-square temperature variance in grid turbulence with linear cross-stream variation in temperature. Symbols, experiment of Alexopoulos & Keffer (1971); —— present prediction, $E = -2{\cdot}0$; - - - - - present prediction, $E = -1{\cdot}955$.

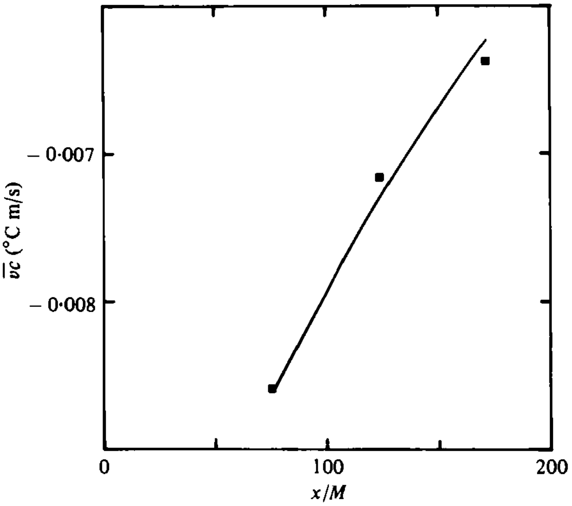


FIGURE 5. Development of cross-stream heat flux in grid turbulence with linear cross-stream variation in temperature. Symbols, experiment of Alexopoulos & Keffer (1971); line, present prediction.

We may obtain further perspective on the quantity $E$ by considering the evolution of $r$ implied by our model equations. By an extension of the procedure employed in section (3.2) we may form an equation for the time scale ratio from our scalar-gradient model equations as

$$\frac{1}{r}\frac{dr}{d\tau} = (\psi - B - 2) + r\left\{(2-D) - \frac{P_c}{\epsilon_c}(2+E)\right\} \tag{3.29}$$

where $P_c = -\overline{u_i c}C_{,i}$ is the rate of production of $\overline{c^2}$. We observe that (3.29) differs from the equation for $r$ derived previously (3.16) by the presence of the term $r(P_c/\epsilon_c)(2+E)$.

The coefficients $B$ and $D$ have been determined in part to match the slow evolution of $r$ in the heated grid flows, and we may usefully adopt a similar approach here. The evolution of $r$ in the Alexopoulos & Keffer (1971) flow was quite slow ($r$ varied by about 5 % over the entire flow which amounted to a turbulence decay interval of $\Delta\tau = 0{\cdot}53$), and it appears reasonable to presume that the evolution of $r$ will be slow in most flows of the type considered here. We see from equation (3.29) that we may ensure that $R$ will evolve at an appropriately slow rate if $E$ is given a value approximately equal to $-2{\cdot}0$. In addition, we may consider the asymptotic behaviour of $r$. Here, however, we have only the values of $r$ for the cases of one-dimensional turbulence and of final-period turbulence to guide us. We see from (3.29) that these two limiting values will be predicted by our scalar model if we specify $E = -2{\cdot}0$. Since we lack further information, we shall adopt the value $-2{\cdot}0$ for $E$ which differs by less than 2·5 % (which is within the accuracy of the data) from the value determined by a least-squares fit to the existing data. We note that with this value for $E$ equation (3.29) becomes identical to equation (3.16) so that the asymptotic behaviour of $r$ will mimic the equilibrium behaviour of $r$ modelled for the case of homogeneous scalar turbulence without mean scalar gradients. In figures 4 and 5 we compare the development of $\overline{c^2}$ and $\overline{vc}$ (the cross-stream heat flux) predicted by our model (using $E = -2{\cdot}0$ and $G = 6{\cdot}6$) with the measured behaviour from the experiment of Alexopoulos & Keffer (1971). In figure 4, we also present for completeness the least-squares-fit prediction curve of $\overline{c^2}$, obtained with $E = -1{\cdot}955$ (the associated curve for $\overline{vc}$ is not presented in figure 5 since it differs negligibly from the prediction curve shown). The agreement between experimental and predicted values is within 2 % for both figures. In addition, we display in figure 2 the simulated and measured decay of $\overline{u^2}/U^2$ which pertains to this experimental flow. Agreement is again seen to be satisfactorily close to the measured values.

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